Semi-Definite Programming for Topology Optimization of Trusses under Multiple Eigenvalue Constraints

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Abstract

Topology optimization problem of trusses for specified eigenvalue of vibration is formulated as Semi-Definite Programming (SDP), and an algorithm is presented based on the Semi-Definite Programming Algorithm (SDPA) which utilizes extensively the sparseness of the matrices. Since the sensitivity coefficients of the eigenvalues with respect to the design variables are not needed, the SDPA is especially useful for the case where the optimal design has multiple fundamental eigenvalues. Global and local modes are defined and a procedure is presented for generating optimal topology from the practical point of view. It is shown in the examples, that SDPA has advantage over existing methods in view of computational efficiency and accuracy of the solutions, and an optimal topology with five-fold fundamental eigenvalue is found without any difficulty.

1 Introduction

The eigenvalues of free vibration as well as the linear buckling load factor are important performance measures of the structures. Therefore there have been many studies for optimization of structures under eigenvalue constraints. It is well known that optimum designs for specified fundamental eigenvalue often have multiple (repeated) eigenvalues. Such an optimal structure was first presented by Olhoff and Rasmussen [1] where necessary conditions for optimality are discussed and an optimal column under buckling constraint is found by using an optimality criteria approach. Masur [2] showed that the necessary conditions by Olhoff and Rasmussen [1] are also sufficient conditions in the case of bimodal optimal solution of symmetric structures. Early developments in this field are summarized in [3].

Difficulties in optimizing distributed parameter structures for specified multiple eigenvalues have been discussed extensively in [4]. It has been shown that the multiple eigenvalues are not differentiable in ordinary sense, and only directional derivatives with respect to the design variables may be calculated. There have been many results presented for bimodal optimal solutions of columns, arches and plates. Bochenek and Gajewski [5] found optimal cross-sectional areas of arches that have at most three-fold eigenvalues of in-plane and out-of-plane buckling. For finite dimensional structures with moderately large number of design variables, however, it is very difficult to find optimum designs for structures with multiple eigenvalues by using conventional approaches of optimality criteria method or mathematical programming.

Several computational approaches have been developed for sensitivity analysis of multiple eigenvalues of finite dimensional structures [6–9]. Khot [10] presented an optimality criteria approach for optimum design of trusses with multiple frequency constraints. Rodriguez *et al.* [11] developed necessary conditions for optimality for problems under constraints on linear buckling load factor, and presented an adjoint variable formulation for sensitivity analysis. Recent developments in this field are summarized in the review paper by Seyranian *et al.* [12]. In spite of theoretical developments for sensitivity analysis of multiple eigenvalues and optimization methods for problems under multiple eigenvalue constraints, no globally convergent algorithm seems to have been presented for optimization of large structures. Nakamura and Ohsaki [13] presented a parametric programming approach to trace a set of optimal solutions under multiple eigenvalue constraints. Although their method has been shown to be effective for bimodal case, it is very difficult to extend it to the problems with larger multiplicity of eigenvalues. In order to overcome the difficulties due to multiplicity of eigenvalues, we present in this paper an algorithm based on the Semi-Definite Programming (SDP) which does not need explicit derivatives of eigenvalues with respect to the design variables.

The SDP is an extension of linear programming in a sense that in addition to linear constraints, it allows the constraints that require matrices to be positive semi-definite (notice that those constraints cannot be expressed as linear constraints). The SDP unifies several convex optimization problems (e.g., linear and quadratic programming) and finds many applications in engineering and combinatorial optimization [14]. Many interior-point methods for linear and quadratic programming have been extended to solve SDPs. As in linear programming, these methods have polynomial time worst-case complexity and perform very well in practice. As a result, SDPs are not much harder to solve than ordinary linear and quadratic programming problems. The SDP has been shown to be effective for topology optimization of trusses considering compliance under given static loads [15, 16], where the maximum value of the compliances among the specified set of loads is minimized under the constraint on total structural volume. Among several softwares for SDPs that are currently available, SDPA (Semi-Definite Programming Algorithm) [17] seems to be fastest. It is a C++ implementation of a Mehrotra-type primal-dual predictor-corrector interior-point method [18, 19] for solving the standard form of SDP. The SDPA incorporates data structures for handling sparse matrices and an efficient method proposed by Fujisawa *et al.* [20] for computing search directions for problems with large sparse matrices.

In this paper, the topology optimization problem for specified eigenvalue of vibration is formulated as SDP, and optimal topologies are computed for several examples of plane and space trusses by applying the SDPA. In the examples, in order to see the effectiveness of the proposed method in view of computational efficiency and accuracy of the solutions, we compare the computational results with those computed by existing parametric programming approach and sequential quadratic programming method. Computational experiments demonstrate that the SDPA computes optimal solutions more accurately and more efficiently than the other two methods. More notably, in the example of a double-layer grid, an optimal solution with five-fold fundamental eigenvalues can be generated without any difficulty, which could not be obtained by the other two methods. These results indicate that the SDPA successfully resolves the computational difficulty that most of the existing methods are faced with in optimization of structures with multiple fundamental eigenvalues.

As is shown in the examples, however, for the most cases there exist secondary members with small cross-sectional areas in the optimal topology, and multiplicity of fundamental eigenvalues should be considered even for a simple truss with small number of members. Those secondary members are needed to prevent the local flexural vibration of a long member formed by straightly linking short members with moderately large cross-sectional areas. From the practical point of view, the optimal topology with secondary members may be very difficult to be constructed. Therefore the secondary members may be removed, if necessary, and the unstable nodes are to be fixed to generate a practically optimal topology [23].

2 Outline of SDPA

Let $R^{n\times n}$ and $\mathcal{S}^n \subset R^{n\times n}$ denote the set of all $n \times n$ real matrices and the set of all $n \times n$ real symmetric matrices, respectively. We use the notation $\mathbf{U} \bullet \mathbf{V}$ for the inner product of $\mathbf{U},\ \mathbf{V} \in R^{n\times n}$, i.e. $\mathbf{U} \bullet \mathbf{V} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{U}_{ij}\mathbf{V}_{ij}$, where $\mathbf{U}_{ij}$ and $\mathbf{V}_{ij}$ denote the (i, j)th element of $\mathbf{U}$ and $\mathbf{V}$, respectively. We write $\mathbf{X} \succeq \mathbf{O}$ and $\mathbf{X} \succ \mathbf{O}$ when $\mathbf{X} \in \mathcal{S}^n$ is positive semi-definite and positive definite, respectively.

Let $\mathbf{F}_i \in \mathcal{S}^n$ $(i = 0, \cdots, m)$, $\mathbf{b} \in R^m$ and $\mathbf{y} \in R^m$. The SDPA solves the standard form SDP and its dual:

$$
\left.\begin{array}{ll}
\mathcal{P}\text{: Minimize } & \mathbf{F}_0 \bullet \mathbf{X}, \\
& \text{subject to } \mathbf{F}_i \bullet \mathbf{X} = b_i \ (i = 1, \cdots, m), \quad \mathbf{X} \in \mathcal{S}^n, \ \mathbf{X} \succeq \mathbf{O}. \\
\mathcal{D}\text{: Maximize } & \sum_{i=1}^{m} b_i y_i, \\
& \text{subject to } \sum_{i=1}^{m} \mathbf{F}_i y_i + \mathbf{Z} = \mathbf{F}_0, \quad \mathbf{Z} \in \mathcal{S}^n, \ \mathbf{Z} \succeq \mathbf{O}.
\end{array}\right\} \tag{1}
$$

For simplicity, we say that $(\mathbf{X}, \mathbf{y}, \mathbf{Z})$ is a feasible solution, an interior-feasible solution, or an optimal solution, respectively, of the SDP (1) if $\mathbf{X}$ is a feasible solution, an interior-feasible solution (i.e., a feasible solution satisfying $\mathbf{X} \succ \mathbf{O}$), or a minimizing solution, respectively, of $\mathcal{P}$ and $(\mathbf{y}, \mathbf{Z})$ is a feasible solution, an interior-feasible solution (i.e., a feasible solution satisfying $\mathbf{Z} \succ \mathbf{O}$), or a maximizing solution, respectively, of $\mathcal{D}$.

In the remainder of this section, we briefly explain the HRVW/KSH/M search direction [18] which we employ in this paper and the algorithmic framework of the SDPA.

2.1 Search direction

In general, the computation of a search direction is a most time-consuming part of computer programs for SDPs including the SDPA. Among many search directions proposed by several groups of researchers [20], we employ the HRVW/KSH/M direction [18] in our numerical experiments because its computation is the cheapest (particularly, for sparse data matrices) when we employ the method proposed by Fujisawa *et al.* [20]. The HRVW/KSH/M direction is the solution $(d\mathbf{X}, d\mathbf{y}, d\mathbf{Z})$ of the system of equations

$$
\left.\begin{array}{l}
\mathbf{F}_i \bullet d\mathbf{X} = p_i \ (i = 1, \cdots, m), \ d\mathbf{X} \in \mathcal{S}^n, \\
\sum_{i=1}^{m} \mathbf{F}_i dy_i + d\mathbf{Z} = \mathbf{D}, \ d\mathbf{Z} \in \mathcal{S}^n, \\
\widehat{d\mathbf{X}}\mathbf{Z} + \mathbf{X} d\mathbf{Z} = \mathbf{K}, \ \widehat{d\mathbf{X}} \in R^{n \times n}, \ d\mathbf{X} = (\widehat{d\mathbf{X}} + \widehat{d\mathbf{X}}^T)/2.
\end{array}\right\} \tag{2}
$$

Here $p_i \in R$, $\mathbf{D} \in \mathcal{S}^n$ and $\mathbf{K} \in R^{n \times n}$ denote a scalar constant, an $n \times n$ constant symmetric matrix, and an $n \times n$ constant matrix, respectively, which are determined by the current point $(\mathbf{X}, \mathbf{y}, \mathbf{Z})$ and some other factors. Note that $\widehat{d\mathbf{X}} \in R^{n \times n}$ serves as an auxiliary variable matrix. Under the linear independence assumption on the set $\{\mathbf{F}_i : i = 1, \cdots, m\}$ of constraint matrices, we know [18] that for any $\mathbf{X} \succ \mathbf{O}$, $\mathbf{Z} \succ \mathbf{O}$, $p_i \in R$ $(i = 1, \cdots, m)$, $\mathbf{D} \in \mathcal{S}^n$, and $\mathbf{K} \in R^{n \times n}$, the system of equations (2) has a unique solution $(d\mathbf{X}, d\mathbf{y}, d\mathbf{Z})$.

We can reduce the system of equations (2) to

$$
\mathbf{B} d\mathbf{y} = \mathbf{b}, \tag{3}
$$

$$
\left.\begin{array}{l}
d\mathbf{Z} = \mathbf{D} - \sum_{i=1}^{m} \mathbf{F}_i dy_i, \\
\widehat{d\mathbf{X}} = (\mathbf{K} - \mathbf{X} d\mathbf{Z})\, \mathbf{Z}^{-1}, \ d\mathbf{X} = (\widehat{d\mathbf{X}} + \widehat{d\mathbf{X}}^T)/2,
\end{array}\right\} \tag{4}
$$

where

$$
\left.\begin{array}{l}
\mathbf{B}_{ij} = \mathbf{X}\mathbf{F}_i \mathbf{Z}^{-1} \bullet \mathbf{F}_j \ (i = 1, \cdots, m, \ j = 1, \cdots, m), \\
b_i = p_i - (\mathbf{K} - \mathbf{X}\mathbf{D})\mathbf{Z}^{-1} \bullet \mathbf{F}_i \ (i = 1, \cdots, m).
\end{array}\right\} \tag{5}
$$

The matrices $\mathbf{X}$, $\mathbf{Z}^{-1}$ and $\mathbf{B}$ are symmetric and dense in general even when all $\mathbf{F}_i$s $(i = 1, \cdots, m)$ are sparse. Hence solving the system of equations (3) in $d\mathbf{y}$ by using a direct method such as the Cholesky factorization and the LDL^T factorization of $\mathbf{B}$ requires $O(m^3)$ arithmetic operations. On the other hand, if we treat all $\mathbf{F}_i$ s $(i = 1, \cdots, m)$ as dense matrices and if we use the above formulae (5) for the coefficient matrix $\mathbf{B}$ in a straightforward way, the computation of $\mathbf{B}$ requires $O(mn^3 + m^2n^2)$ arithmetic operations. Therefore computing the coefficient matrix $\mathbf{B}$ is more crucial than solving $\mathbf{B}d\mathbf{y} = \mathbf{b}$ in the entire computation of the HRVW/KSH/M direction.

Fujisawa *et al.* [20] proposed three distinct formulae $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ for computing $\mathbf{B}$, and their efficient combination $\mathcal{F}_*$. They demonstrated through numerical experiments that the combined formula $\mathcal{F}_*$ worked very efficiently when some of $\mathbf{F}_i$ s $(i = 1, \cdots, m)$ are sparse. We incorporated their formula $\mathcal{F}_*$ into SDPA. See [20] for more details.

2.2 The algorithmic framework of SDPA

Step 0 : Set an initial point $(\mathbf{X}^0, \mathbf{y}^0, \mathbf{Z}^0)$ with $\mathbf{X}^0 \succ \mathbf{O}, \mathbf{Z}^0 \succ \mathbf{O}$. Decide on the search direction to use. Set the parameters: $0.0 < \epsilon^*$, $1 < \omega^*$, $0.01 \leq \beta^* \leq 0.10$ and $\beta^* \leq \bar{\beta} \leq 0.20$ (The default values of these parameters are: $\epsilon^* = 1.0 \times 10^{-8}$, $\omega^* = 2.0$, $\beta^* = 0.05$ and $\bar{\beta} = 0.1$). Let $k = 0$.

Step 1 : If the current iterate $(\mathbf{X}^k, \mathbf{y}^k, \mathbf{Z}^k)$ is feasible and the relative gap

$$\frac{|P - D|}{\max\{1.0,\ (|P| + |D|)/2\}}$$

is smaller than ϵ^*, then stop the iteration. Here P and D denote the primal and the dual objective values, respectively. If we detect that there is no feasible solution $(\mathbf{X}, \mathbf{y}, \mathbf{Z})$ such that $\omega^*\mathbf{X}^0 \succeq \mathbf{X} \succeq \mathbf{O}$ and $\omega^*\mathbf{Z}^0 \succeq \mathbf{Z} \succeq \mathbf{O}$, then stop the iteration. See [18] for details on how to find such information on infeasibility.

Step 2 : (Predictor Step) Let

$$\beta_p = \begin{cases} 0 & \text{if the current iterate is feasible,} \\ \bar{\beta} & \text{otherwise.} \end{cases}$$

Solve the system of equations (2) with $\mathbf{K} = \beta_p(\mathbf{X} \bullet \mathbf{Z}/n)\mathbf{I} - \mathbf{XZ}$ to compute the predictor direction $(d\mathbf{X}_p, d\mathbf{y}_p, d\mathbf{Z}_p)$.

Step 3 : (Corrector Step) Let

$$\beta = \frac{(\mathbf{X} + \bar{\alpha}_p d\mathbf{X}_p) \bullet (\mathbf{Z} + \bar{\alpha}_d d\mathbf{Z}_p)}{(\mathbf{X} \bullet \mathbf{Z})},$$

where $\bar{\alpha}_p$ and $\bar{\alpha}_d$ are computed as in [21]. Choose the parameter β_c as follows:

$$\beta_c = \begin{cases} \max\{\beta^*, \beta^2\} & \text{if the current iterate is feasible and } \beta \leq 1.0, \\ \max\{\bar{\beta}, \beta^2\} & \text{if the current iterate is infeasible and } \beta \leq 1.0, \\ 1.0 & \text{otherwise.} \end{cases}$$

Compute the corrector direction $(d\mathbf{X}_c, d\mathbf{y}_c, d\mathbf{Z}_c)$ by solving the system of equations (2) with $\mathbf{K} = \beta_c(\mathbf{X} \bullet \mathbf{Z}/n)\mathbf{I} - \mathbf{XZ} - d\mathbf{X}_p d\mathbf{Z}_p$.

Step 4 : Set the next iterate $(\mathbf{X}^{k+1}, \mathbf{y}^{k+1}, \mathbf{Z}^{k+1})$ such that

$$\mathbf{X}^{k+1} = \mathbf{X}^k + \alpha_p d\mathbf{X}_c \ \text{ and } (\mathbf{y}^{k+1}, \mathbf{Z}^{k+1}) = (\mathbf{y}^k, \mathbf{Z}^k) + \alpha_d(d\mathbf{y}_c, d\mathbf{Z}_c),$$

where α_p and α_d are computed as in [21].

Step 5 : $k \longleftarrow k + 1$ and go to Step 1.

3 Formulation of topology optimization problem by SDP

Consider a truss with fixed locations of nodes and members that can exist. The vector of member cross-sectional areas is denoted by $\mathbf{A} = \{A_i\}$. Let $\mathbf{K}$ and $\mathbf{M}_s$ denote the stiffness matrix and the mass matrix due to the structural mass both of which are functions of $\mathbf{A}$. The mass matrix for nonstructural mass is denoted by $\mathbf{M}_0$.

The eigenvalue problem of vibration is formulated as

$$\mathbf{K}\Phi_r = \Omega_r(\mathbf{M}_s + \mathbf{M}_0)\Phi_r \ (r = 1, 2, \cdots, N^d), \tag{6}$$

where Ω_r and Φ_r are the rth eigenvalue and eigenvector, respectively, and N^d is the number of freedom of displacements. The eigenvector Φ_r is normalized by

$$\Phi_r^T(\mathbf{M}_s + \mathbf{M}_0)\Phi_r = 1 \ (r = 1, 2, \cdots, N^d). \tag{7}$$

Let $\bar{\Omega}$ denote the specified lower bound of the eigenvalues. The topology optimization problem for specified fundamental eigenvalue is formulated as

$$\left.\begin{array}{lll} \text{TOP:} & \text{Minimize} & \displaystyle\sum_{i=1}^{N^m} A_i L_i, \\ & \text{subject to} & \Omega_r \geq \bar{\Omega} \ (r = 1, 2, \cdots, N^d), \\ & & A_i \geq 0 \ (i = 1, 2, \cdots, N^m), \end{array}\right\} \tag{8}$$

where L_i is the length of the ith member, and N^m is the number of members. The optimal topology is found by removing the members with $A_i = 0$. A small positive lower bound is usually given for A_i throughout the optimization process in order to prevent instability of the structure.

If the fundamental eigenvalue of the optimum design is simple, TOP may easily be solved by using a nonlinear programming or an optimality criteria approach [24, 25], because there is no difficulty in calculating the sensitivity coefficients of Ω_1 with respect to A_i. In the case of multiple eigenvalues, only directional sensitivity coefficients can be calculated [4]. Although some formulations of sensitivity analysis of multiple eigenvalues have been presented [7, 12], it is not clear if those formulations can be used for optimizing large structures. In this paper, we convert the generalized eigenvalue problem (6) into a standard form of SDP (1) and solve it by using SDPA.

Consider a structure where $\Omega_1 \geq \bar{\Omega}$ is satisfied. In this case the Rayleigh's principle leads to the following inequality for any kinematically admissible mode $\boldsymbol{\psi}$:

$$\boldsymbol{\psi}^T[\mathbf{K} - \bar{\Omega}(\mathbf{M}_s + \mathbf{M}_0)]\boldsymbol{\psi} \geq 0. \tag{9}$$

This inequality implies that the matrix $\{\mathbf{K} - \bar{\Omega}(\mathbf{M}_s + \mathbf{M}_0)\}$ is positive semi-definite, and formulations of SDP may be possible. The matrices $\mathbf{K}_i$ and $\mathbf{M}_i$ are defined as

$$\mathbf{K}_i = \frac{\partial \mathbf{K}}{\partial A_i}, \quad \mathbf{M}_i = \frac{\partial \mathbf{M}_s}{\partial A_i}. \tag{10}$$

Since $\mathbf{K}$ and $\mathbf{M}_s$ are linear functions of A_i for trusses, those are written as

$$\mathbf{K} = \sum_{i=1}^{N^m} A_i \mathbf{K}_i, \quad \mathbf{M}_s = \sum_{i=1}^{N^m} A_i \mathbf{M}_i. \tag{11}$$

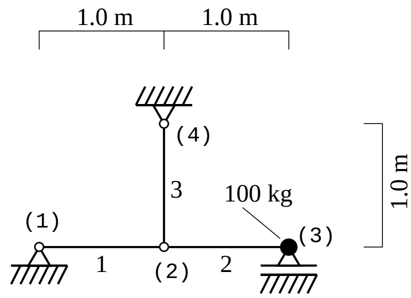

Figure 1: A three-bar pin-jointed truss.

The primal and dual problems of SDP for this case are formulated as

$$\left.\begin{array}{lll} \mathcal{P}': & \text{Minimize} & \displaystyle\sum_{i=1}^{m} A_i L_i, \\ & \text{subject to} & \mathbf{X} = \displaystyle\sum_{i=1}^{m} (\mathbf{K}_i - \bar{\Omega}\mathbf{M}_i) A_i - \bar{\Omega}\mathbf{M}_0, \\ & & \mathbf{X} \in \mathcal{S}^n, \mathbf{X} \succeq \mathbf{O}, \\ & & A_i \geq 0 \ (i = 1, 2, \cdots, N^m). \end{array}\right\} \quad (12)$$

$$\left.\begin{array}{lll} \mathcal{D}': & \text{Maximize} & \bar{\Omega}\mathbf{M}_0 \bullet \mathbf{Y}, \\ & \text{subject to} & (\mathbf{K}_i - \bar{\Omega}\mathbf{M}_i) \bullet \mathbf{Y} \leq L_i, \\ & & \mathbf{Y} \in \mathcal{S}^n, \mathbf{Y} \succeq \mathbf{O}, \end{array}\right\} \quad (13)$$

where $m = N^m$ and $n = N^d$. Notice here that the roles of $\mathcal{P}$ and $\mathcal{D}$ of standard formulation of SDP are exchanged in the above description of $\mathcal{P}'$ and $\mathcal{D}'$ since the original problem we are concerned with is written in the form of $\mathcal{P}'$. Problems $\mathcal{P}'$ and $\mathcal{D}'$ are solved successively to find optimal solutions by using the SDPA algorithm. It is important to note here that sensitivity coefficients of eigenvalues with respect to the design variables are not needed in the optimization process. Therefore there is no difficulty, as will be shown in the examples, in finding the solutions with multiple fundamental eigenvalues.

It is well known, however, that optimal pin-jointed trusses often turn out to be unstable. For the case of stress and displacement constraints, a small lower bound $\bar{A}_i$ is given for A_i, and those members with $A_i = \bar{A}_i$ is removed from the optimal design. Then the unstable nodes are fixed to generate a stable truss of an optimal topology. Kirsch [22] presented a method for finding an optimal topology by successively solving linear programming problems while the compatibility conditions for strains are neglected. For the case of frequency constraints, however, the instability leads to a difficulty due to multiplicity of the fundamental eigenvalues in the optimal topology [13]. It is discussed in the following that formulation of multiple eigenvalues is needed even for a simple three-bar truss as shown in Fig. 1.

Consider a case where lower bounds are given for A_i as

$$A_i \geq \xi \bar{A}_i^0 \ (i = 1, 2, \cdots, N^m), \quad (14)$$

where $\bar{A}_i^0$ is a constant value and ξ is a parameter. In [13], the optimal solutions are traced by decreasing the parameter ξ, and the optimal truss corresponding to $\xi = 0$ is found by extrapolating the solutions at a sufficiently small value of ξ. This approach is referred to as *parametric programming approach* in the examples. The initial value ξ^0 for ξ is given so that Ω_1 of the truss with $A_i = \bar{A}_i = \xi^0 \bar{A}_i^0$ for all the members is equal to the specified value. If the nonstructural mass at node 3 is sufficiently large compared with the total structural mass, the lowest eigenmode at the initial design is such that the nodes 2 and 3 moves horizontally, and the axial deformation of member 3 is negligibly small compared with those of members 1 and 2. This type of mode associated with vibration of nonstructural masses is referred to as *global mode* in the following.

Since the axial deformation of member 3 is negligibly small, A_3 $(= \xi \bar{A}_3^0)$ decreases as ξ is decreased. It is obvious, however, that the pin-jointed truss is unstable if member 3 is removed, and the fundamental eigenmode is such that the vibration of nonstructural mass is negligibly small. This type of mode is referred to as *local mode* in the following. Consequently, there exists a member with extremely small cross-sectional area and two lowest eigenvalues coincide in the *theoretically optimal truss*.

This result suggests that the formulations of multiple eigenvalues are for most cases necessary to find an optimal topology of a pin-jointed truss, and secondary members with small cross-sectional areas will exist in the optimal topology. From the practical point of view, however, the optimal topology with secondary members are not needed, and the designers are not interested in the local mode which is simply suppressed by adding flexural stiffness at the joints. Therefore a *practically optimal topology* may be generated by removing the secondary members and by fixing the unstable nodes as node 3 in Fig. 1. Note that the fundamental eigenvalue of the practically optimal truss is simple in this example.

Our purpose here is to compare the performances of SDPA with other existing method in the view of accuracy and computational efficiency. Therefore the results are discussed before the secondary members are removed. It may be straightforward to generate practical designs if necessary. A stable optimal topology may also be found by allowing the members to exist between all the possible pairs of the nodes. In this case, however, the number of design variables is very large, and substantial computational effort will be needed.

4 Examples

Optimal topologies are computed for plane and space trusses by the proposed method using SDPA as well as Parametric Programming (PP) and Sequential Quadratic Programming (SQP) [26], in order to compare the computational efficiency and accuracy of the results among the three methods. In the following examples, the material of the members is steel where elastic modulus E is 205.8 GPa and the mass density ρ is 7.86×10^{-3} kg/cm^2. In SDPA, E and ρ are scaled so that $E = 1000.0$ is satisfied to prevent divergence in the process of finding a feasible solution. The specified eigenvalue is 1000.0 rad^2/s^2 for all the cases. The computation has been carried out on Sun Ultra II (Ultra SPARC II 300MHz with 256 MB memory).

4.1 Plane square grids

Optimal topologies are found for plane square trusses with 2×2, 3×3, 4×4 and 5×5 grids to compare the performances of the methods. A nonstructural mass of 2.1×10^4 kg is located at the the upper-right corner for each case. A 5×5 grid is as shown in Fig. 2, where L_i for all the vertical and horizontal members are 200.0 cm. The optimal topology of 5×5 grid found by SDPA after removing extremely slender members with $A_i < 2.0 \times 10^{-3}$ cm^2 is as shown in Fig. 3, where the width of each member is proportional to its cross-sectional area. Note from Fig. 3 that there exists a kind of net with secondary members for preventing instability of the ten-bar truss formed by the primal members with moderately large cross-sectional areas. Those secondary members cannot be removed because the two long members, each composed of five short members, will be unstable without those members.

The multiplicity of the lowest eigenvalues is two, and the corresponding modes are as illustrated in Fig. 4. It may be observed from Fig. 4 that the displacements of node 9 where the nonstructural mass is located is very large in the mode (a), whereas local flexural deformation at node 1 dominates in the mode (b). The local modes such as mode (b) may be suppressed by fixing the unstable nodes 1-8. The maximum and minimum values of the cross-sectional areas of the primary members are 43.991 cm^2 and 40.566 cm^2, respectively, whereas those of the secondary members are 2.2299

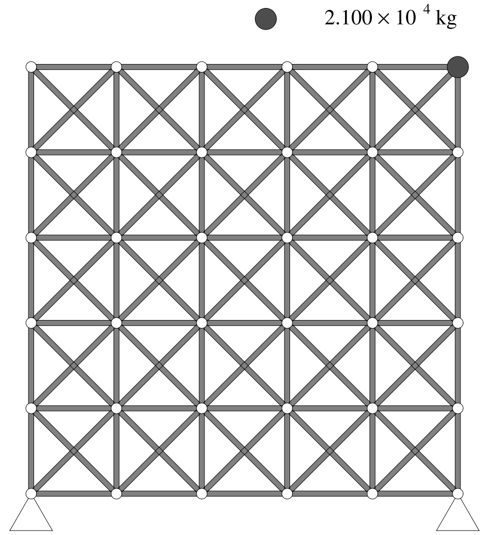

Figure 2: A 5 × 5 plane square grid.

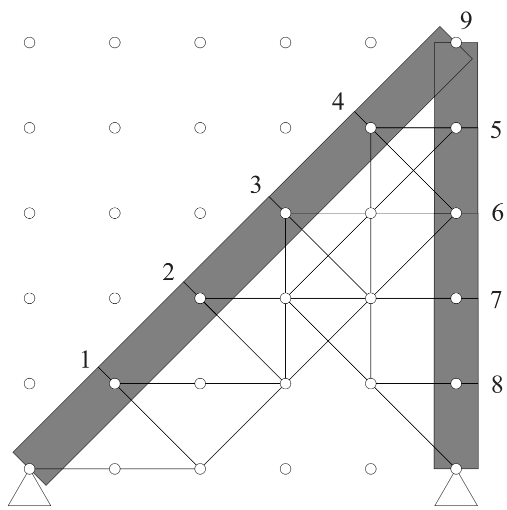

Figure 3: Optimal topology of 5 × 5 grid.

cm^2 and 6.7598×10^{-3} cm^2, respectively. A practically optimal topology may be found, if necessary, by removing the secondary members and fixing the unstable nodes 1-8 in Fig. 3 to generate a frame with two members. Note that the node 9 is not fixed. Let γ denote the radius of gyration of each member. The fundamental eigenvalue of the frame for $\gamma = 30.0$ cm is 973.28 rad^2/s^2 which is smaller than the specified value due to the lateral deformation of two long members. The eigenvalue may be increased by assigning larger value for γ.

The results by SDPA, PP and SQP are listed in Table 1. The optimal solutions for 2×2, 3×3, and 4×4 are as shown in Figs. 5-7, respectively. It may be observed from these results that the performance of SDPA is better than that of PP in view of accuracy, and CPU time of SDPA is less than that of SQP. In addition to these advantages, SDPA has no difficulty in finding optimal solutions with multiple eigenvalues. Note that the difference among the second eigenvalues computed by three methods is very large, because those are sensitive to the cross-sectional areas of the secondary members. The second eigenvalues, however, are associated with local modes which are not practically important.

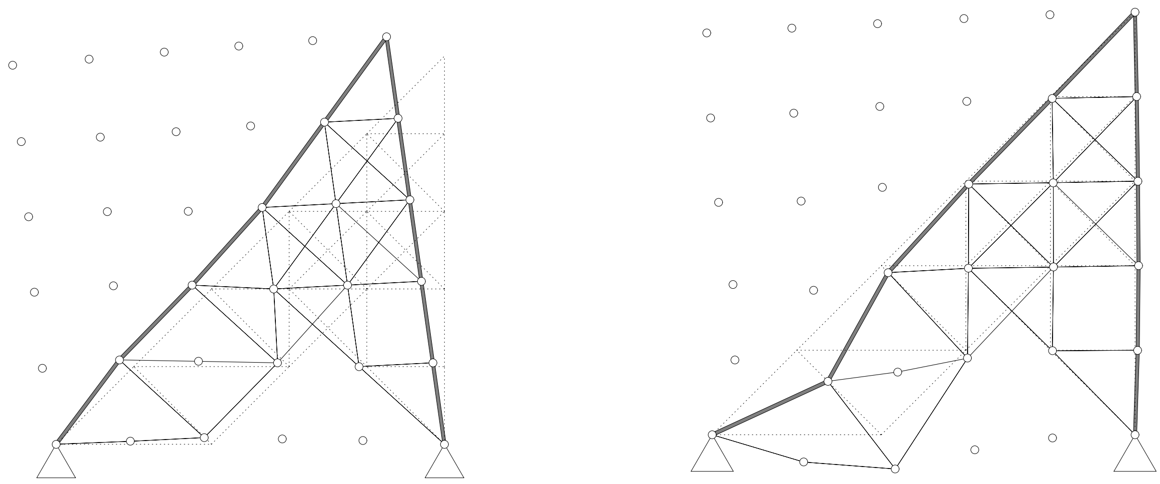

Figure 4: Eigenmodes of optimal 5×5 grid.

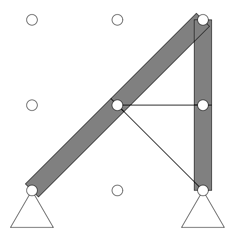

Figure 5: Optimal topology of 2×2 grid.

Since formulation of multiple eigenvalues has not been used for SQP, the optimization process has not converged if $\bar{A}_i = 0.01$ for the 5×5 grid. Therefore moderately large lower bound is needed for SQP to prevent the divergence due to the multiplicity of eigenvalues. Note again that it is not important from the practical point of view to find optimal solutions with multiple eigenvalues one of which is associated with a locally vibrating mode such as mode (b) in Fig. 4. Since CPU time for SQP will be much larger if the multiplicity of the fundamental eigenvalues is considered, the efficiency of SDPA compared with SQP has been successfully demonstrated by these examples. Positive lower

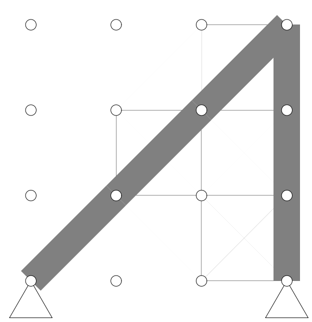

Figure 6: Optimal topology of 3×3 grid.

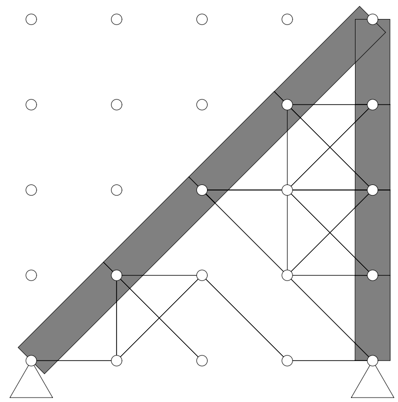

Figure 7: Optimal topology of 4×4 grid.

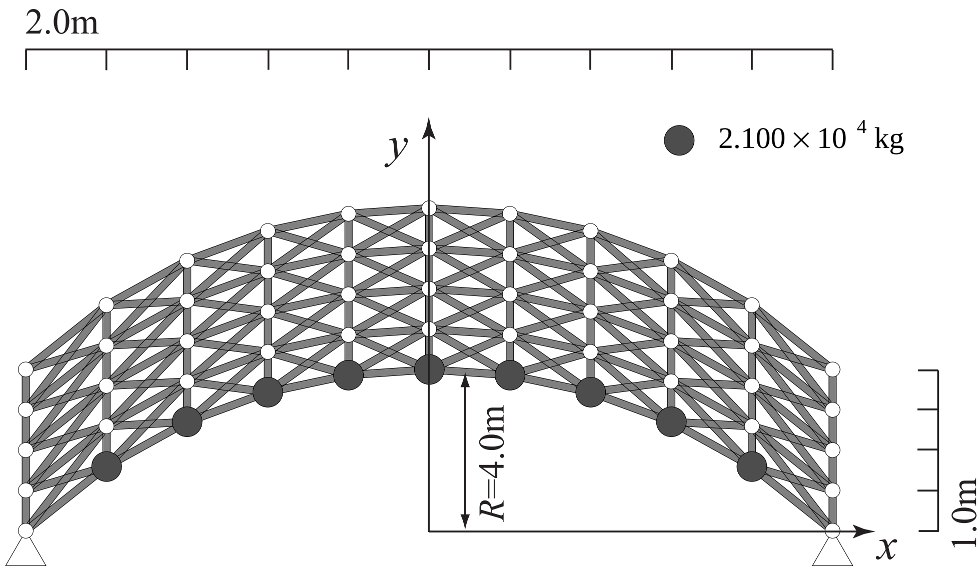

Figure 8: A plane arch grid.

bound on cross-sectional areas are also given for PP to avoid unnecessary computational cost due to multiplicity of eigenvalues corresponding to locally vibrating modes.

4.2 A plane arch grid

Consider next a plane arch grid as shown in Fig. 8. Nonstructural masses are located at the nodes along the lowest circle. The optimal topology found by SDPA after removing the members with $A_i < 2.0 \times 10^{-3}$ cm^2 is as shown in Fig. 9. The results by SDPA and PP are also listed in Table 1. For PP, additional constraints for the cross-sectional areas are given in order to ensure that only symmetric designs are produced. On the other hand, a symmetric solution has been automatically found by SDPA without requiring such additional constraints.

The multiplicity of the fundamental eigenvalues of the optimal truss is two. The fundamental eigenmodes are symmetric and antisymmetric, respectively, as shown in Fig. 10, with respect to the y-axis in Fig. 8, and are the global modes with no significant local bending deformation. Since the solution has multiple eigenvalues, SQP is not executed for this truss, and thus the result is not shown in Table 1. It should be noted here that the CPU time of PP is very large compared with that of SDPA, because the number of linear equations to be solved simultaneously in PP dramatically increases as the multiplicity of the fundamental eigenvalue is increased.

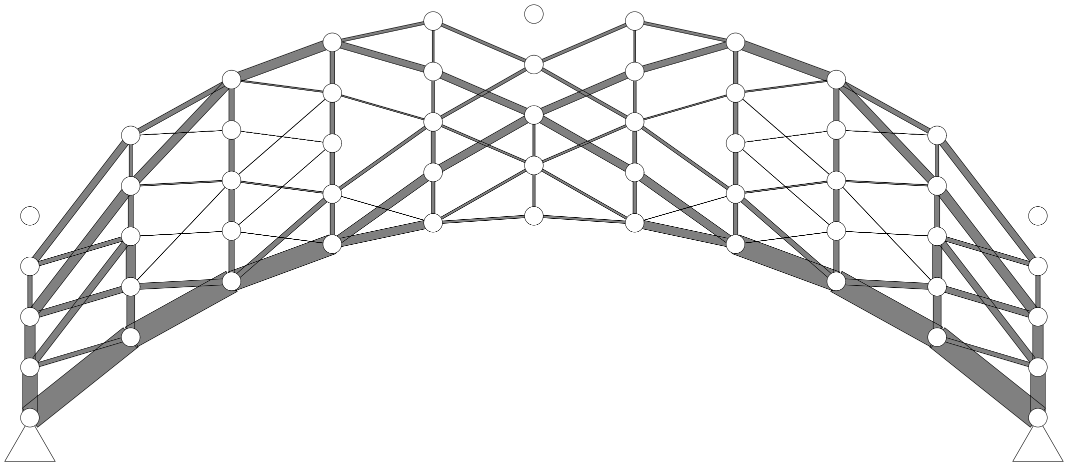

Figure 9: Optimal topology of the plane arch grid.

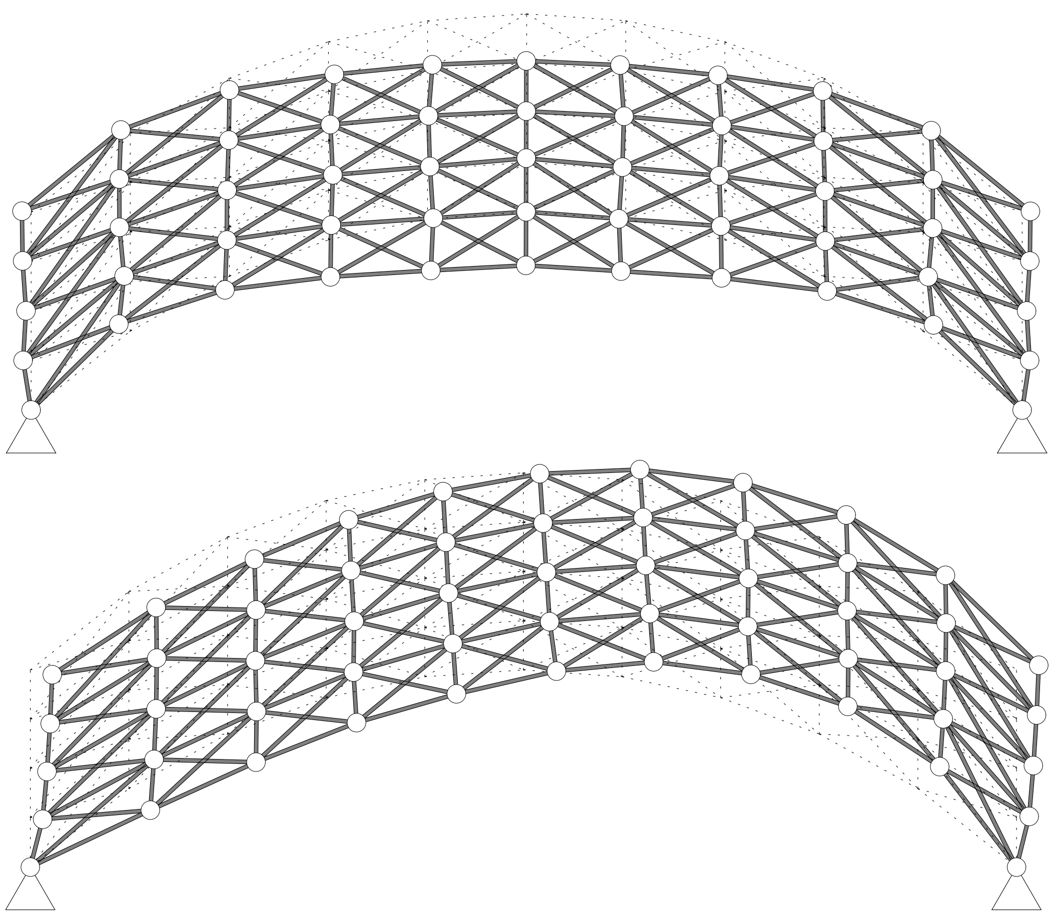

Figure 10: Eigenmodes of optimal arch grid.

Table 1: Comparison of performances of SDPA, PP and SQP.

		SDPA [20]	PP [23]	SQP [26]
Plane square grid 2×2 ($N^m = 20$) ($N^d = 14$)	Volume (cm^3)	1.6355×10^4	1.6368×10^4	1.6357×10^4
	Ω_1 (rad^2/s^2)	1000.0	999.55	1000.0
	Ω_2 (rad^2/s^2)	2145.3	5977.8	2000.3
	$\bar{A}_i$ (cm^2)	0.0	0.01	0.001
	CPU (s)	0.10	0.83	1.21
Plane square grid 3×3 ($N^m = 42$) ($N^d = 28$)	Volume (cm^3)	3.6886×10^4	3.6905×10^4	3.6890×10^4
	Ω_1 (rad^2/s^2)	1000.0	999.44	1000.0
	Ω_2 (rad^2/s^2)	1045.7	2531.6	1011.9
	$\bar{A}_i$ (cm^2)	0.0	0.01	0.001
	CPU (s)	0.43	2.65	5.93
Plane square grid 4×4 ($N^m = 72$) ($N^d = 46$)	Volume (cm^3)	6.5776×10^4	6.6126×10^4	6.5841×10^4
	Ω_1 (rad^2/s^2)	1000.0	999.49	1000.0
	Ω_2 (rad^2/s^2)	1005.0	4551.5	1903.0
	$\bar{A}_i$ (cm^2)	0.0	0.05	0.01
	CPU (s)	1.67	5.76	15.96
Plane square grid 5×5 ($N^m = 110$) ($N^d = 68$)	Volume (cm^3)	1.0320×10^5	1.0371×10^5	1.0446×10^5
	Ω_1 (rad^2/s^2)	1000.0	999.39	1000.0
	Ω_2 (rad^2/s^2)	1000.0	3052.9	5219.3
	Multiplicity	2	1	1
	$\bar{A}_i$ (cm^2)	0.0	0.01	0.1
	CPU (s)	4.50	14.76	44.31
A plane arch grid ($N^m = 174$) ($N^d = 106$)	Volume (cm^3)	6.4493×10^5	6.4497×10^5	
	Ω_1 (rad^2/s^2)	1000.0	998.93	
	Ω_2 (rad^2/s^2)	1000.0	999.42	
	Multiplicity	2	2	
	$\bar{A}_i$ (cm^2)	0.0	0.01	
	CPU (s)	15.04	382.09	
Double-layer grid ($N^m = 128$) ($N^d = 111$)	Volume (cm^3)	8.7110×10^5		
	$\Omega_1, \cdots, \Omega_5$ (rad^2/s^2)	1000.0		
	Multiplicity	5		
	$\bar{A}_i$ (cm^2)	0.0		
	CPU (s)	13.52		

Table 2: Symmetricity of the fundamental eigenmodes of the double-layer grid.

	xz-plane	yz-plane
Mode 1	S	S
Mode 2	A	S
Mode 3	A	S
Mode 4	S	A
Mode 5	A	A

4.3 A double-layer grid

Consider next a double-layer grid as shown in Fig. 11. Nonstructural masses are located at all the upper nodes. The lengths of members in x- and y-directions are 300.0 cm and 200.0 cm, respectively, and the distance between the upper and lower planes is 200.0 cm. The truss has two planes of symmetry. The optimal topology found by SDPA after removing members with $A_i < 2.0 \times 10^{-3}$ cm^2 is as shown in Fig. 12. The optimization results by SDPA is as listed in Table 1. Note that the values of five lowest eigenvalues are all equal to 1000.0 rad^2/s^2, i.e. the multiplicity of eigenvalues of the optimal solution is five, where all the eigenmodes are global modes. Symmetricity properties of the five fundamental eigenmodes are as listed in Table 2, where S and A indicate symmetric and

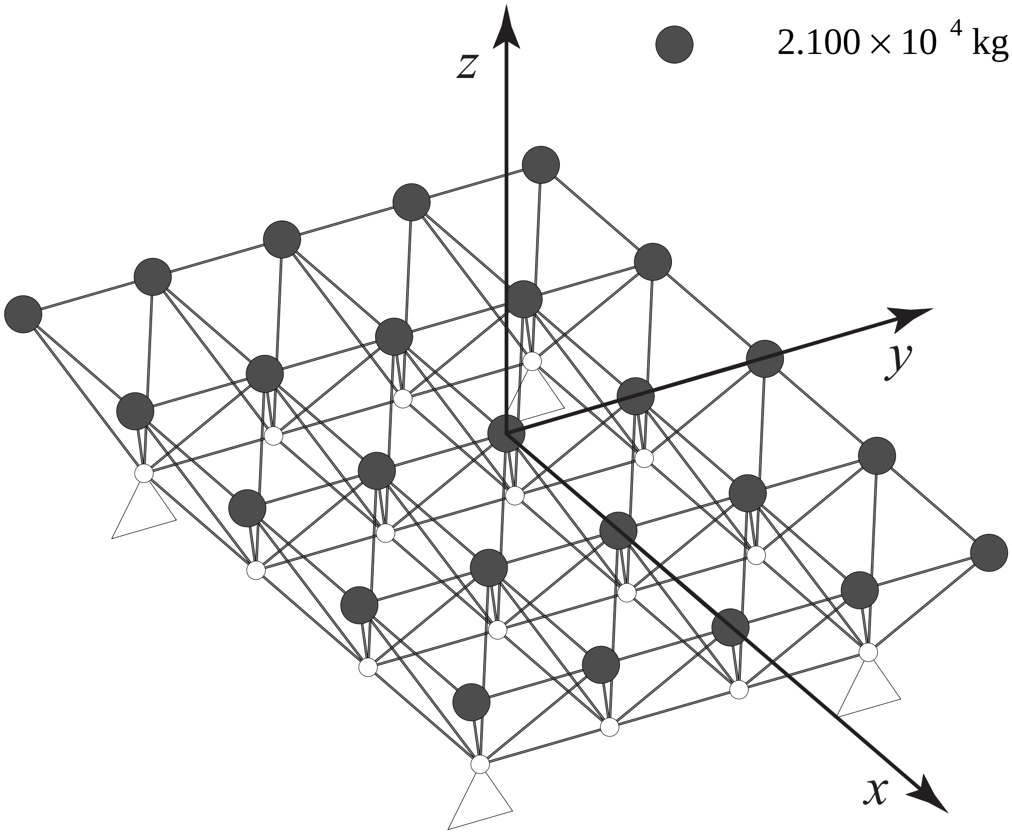

Figure 11: A double-layer grid.

antisymmetric, respectively. SDPA has not found any difficulty in computing an optimal solution even for such case with five-fold fundamental eigenvalues. An extra benefit of using SDPA is that the symmetric solution has been automatically found for this case without imposing any additional constraints.

5 Conclusions

The optimum design problem of trusses under constraints on fundamental eigenvalue of vibration has been formulated as a Semi-Definite Programming (SDP), and an algorithm has been presented for topology optimization. The proposed algorithm based on the Semi-Definite Programming Algorithm (SDPA) is very effective for the case of optimum designs with multiple eigenvalues, because sensitivity coefficients of the eigenvalues with respect to the design variables are not needed and optimal solutions are found without any modification of the algorithm.

Since SDPA fully utilizes sparseness of the matrices, computational cost is very small compared with those of parametric programming approach and sequential quadratic programming algorithm. In the examples, an optimum design with at most five-fold fundamental eigenvalues has been found without any difficulty. Note that no significant increase seems to be observed in CPU time as a result of multiplicity of eigenvalues. In addition to these advantages, a symmetric solution is found without assigning any side constraints in order to preserve symmetricity of the cross-sectional areas.

It has been shown in the examples of plane square grids that a kind of net formed by the secondary members are generated in the optimal topology to prevent the lateral vibration of long members. A practically optimal topology may be found by removing those secondary members and

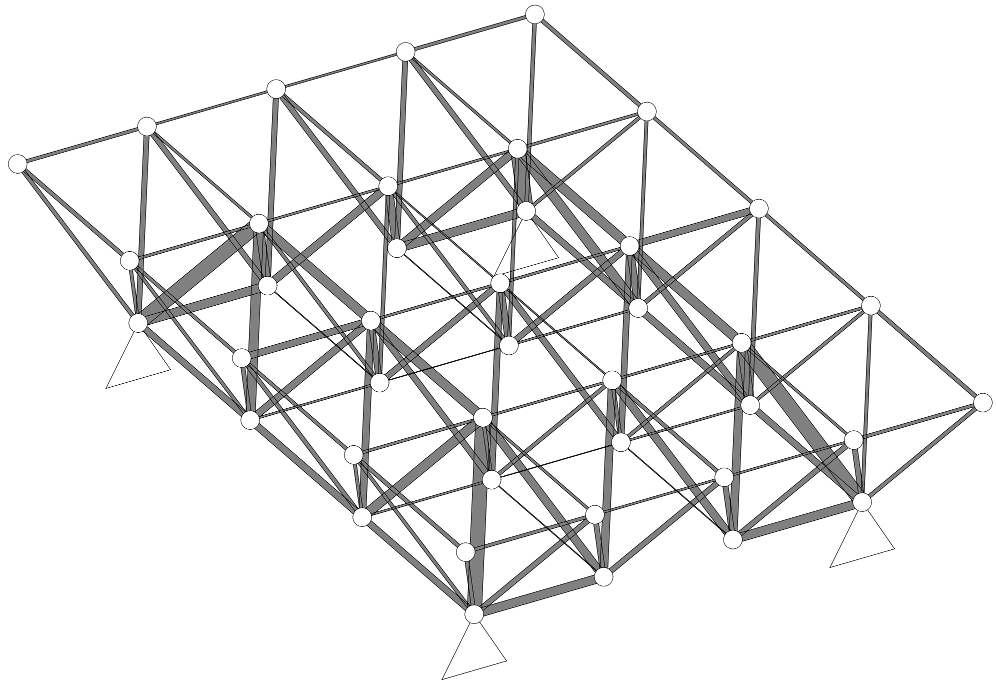

Figure 12: Optimal topology of a double-layer grid.

by fixing the unstable nodes to generate long members with moderately large flexural stiffness.

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